# Inventory of continuous and discrete distributions provided in **actuar**

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## 1 Introduction

R includes functions to compute the probability density function (pdf) or the probability mass function (pmf), the cumulative distribution function (cdf) and the quantile function, as well as functions to generate variates from a fair number of continuous and discrete distributions. For some root `foo`, the support functions are named `dfoo`, `pfoo`, `qfoo` and `rfoo`, respectively.

Package **actuar** provides `d`, `p`, `q` and `r` functions for a large number of continuous distributions useful for loss severity modeling; for phase-type distributions used in computation of ruin probabilities; for zero-truncated and zero-modified extensions of the discrete distributions commonly used in loss frequency modeling; for the heavy tailed Poisson-inverse Gaussian discrete distribution. The package also introduces support functions to compute raw moments, limited moments and the moment generating function (when it exists) of continuous distributions.

## 2 Additional continuous distributions

The package provides support functions for all the probability distributions found in Appendix A of Klugman et al. (2012) and not already present in base R, excluding the log-$t$, but including the loggamma distribution (Hogg and Klugman, 1984). These distributions mostly fall under the umbrella of extreme value or heavy tailed distributions.

Table 1 lists the distributions supported by **actuar** — using the nomenclature of Klugman et al. (2012) — along with the root names of the R functions.

| Family | Distribution | Root |
|---|---|---|
| Transformed beta | Transformed beta | `trbeta` |
| | Burr | `burr` |
| | Loglogistic | `llogis` |
| | Paralogistic | `paralogis` |
| | Generalized Pareto | `genpareto` |
| | Pareto | `pareto` |
| | Inverse Burr | `invburr` |
| | Inverse Pareto | `invpareto` |
| | Inverse paralogistic | `invparalogis` |
| Transformed gamma | Transformed gamma | `trgamma` |
| | Inverse transformed gamma | `invtrgamma` |
| | Inverse gamma | `invgamma` |
| | Inverse Weibull | `invweibull` |
| | Inverse exponential | `invexp` |
| Other | Loggamma | `lgamma` |
| | Gumbel | `gumbel` |
| | Inverse Gaussian | `invgauss` |
| | Single parameter Pareto | `pareto1` |
| | Generalized beta | `genbeta` |

Table 1: Probability distributions supported by **actuar** classified by family and root names of the R functions.

Appendix A details the formulas implemented and the name of the argument corresponding to each parameter. By default, all functions (except those for the Pareto distribution) use a rate parameter equal to the inverse of the scale parameter. This differs from Klugman et al. (2012) but is better in line with the functions for the gamma, exponential and Weibull distributions in base R.

In addition to the `d`, `p`, `q` and `r` functions, **actuar** introduces `m`, `lev` and `mgf` functions to compute, respectively, the theoretical raw moments

$$m_k = E[X^k],$$

the theoretical limited moments

$$E[(X \wedge x)^k] = E[\min(X, x)^k]$$

and the moment generating function

$$M_X(t) = E[e^{tX}],$$

when it exists. Every distribution of Table 1 is supported, along with the following distributions of base R: beta, exponential, chi-square, gamma, lognormal, normal (no `lev`), uniform and Weibull.

The m and lev functions are especially useful for estimation methods based on the matching of raw or limited moments; see the 'lossdist' vignette for their empirical counterparts. The mgf functions come in handy to compute the adjustment coefficient in ruin theory; see the 'risk' vignette.

## 3 Support for phase-type distributions

In addition to the 19 distributions of Table 1, the package provides support for a family of distributions deserving a separate presentation. Phase-type distributions (Neuts, 1981) are defined as the distribution of the time until absorption of continuous time, finite state Markov processes with $m$ transient states and one absorbing state. Let

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{0} & 0 \end{bmatrix} \tag{1}$$

be the transition rates matrix (or intensity matrix) of such a process and let $(\boldsymbol{\pi}, \pi_{m+1})$ be the initial probability vector. Here, $\boldsymbol{T}$ is an $m \times m$ non-singular matrix with $t_{ii} < 0$ for $i = 1, \dots, m$ and $t_{ij} \geq 0$ for $i \neq j$, $\boldsymbol{t} = -\boldsymbol{T}\boldsymbol{e}$ and $\boldsymbol{e}$ is a column vector with all components equal to 1. Then the cdf of the time until absorption random variable with parameters $\boldsymbol{\pi}$ and $\boldsymbol{T}$ is

$$F(x) = \begin{cases} \pi_{m+1}, & x = 0, \\ 1 - \boldsymbol{\pi} e^{\boldsymbol{T}x} \boldsymbol{e}, & x > 0, \end{cases} \tag{2}$$

where

$$e^{\boldsymbol{M}} = \sum_{n=0}^{\infty} \frac{\boldsymbol{M}^n}{n!} \tag{3}$$

is the matrix exponential of matrix $\boldsymbol{M}$.

The exponential distribution, the Erlang (gamma with integer shape parameter) and discrete mixtures thereof are common special cases of phase-type distributions.

The package provides d, p, r, m and mgf functions for phase-type distributions. The root is phtype and parameters $\boldsymbol{\pi}$ and $\boldsymbol{T}$ are named prob and rates, respectively; see also Appendix B.

For the package, function pphtype is central to the evaluation of the ruin probabilities; see '?ruin' and the 'risk' vignette.

## 4 Extensions to standard discrete distributions

The package introduces support functions for counting distributions commonly used in loss frequency modeling. A counting distribution is a discrete distribution defined on the non-negative integers $0, 1, 2, \dots$.

Let $N$ be the counting random variable. We denote $p_k$ the probability that the random variable $N$ takes the value $k$, that is:

$$p_k = \Pr[N = k].$$

Klugman et al. (2012) classify counting distributions in two main classes. First, a discrete random variable is a member of the $(a, b, 0)$ class of distributions if there exists constants $a$ and $b$ such that

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \quad k = 1, 2, \dots.$$

The probability at zero, $p_0$, is set such that $\sum_{k=0}^{\infty} p_k = 1$. The members of this class are the Poisson, the binomial, the negative binomial and its special case, the geometric. These distributions are all well supported in base R with `d`, `p`, `q` and `r` functions.

The second class of distributions is the $(a, b, 1)$ class. A discrete random variable is a member of the $(a, b, 1)$ class of distributions if there exists constants $a$ and $b$ such that

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \quad k = 2, 3, \dots.$$

One will note that recursions start at $k = 2$ for the $(a, b, 1)$ class. Therefore, the probability at zero can be any arbitrary number $0 \leq p_0 \leq 1$.

Setting $p_0 = 0$ defines a subclass of so-called *zero-truncated* distributions. The members of this subclass are the zero-truncated Poisson, the zero-truncated binomial, the zero-truncated negative binomial and the zero-truncated geometric.

Let $p_k^T$ denote the probability mass in $k$ for a zero-truncated distribution. As above, $p_k$ denotes the probability mass for the corresponding member of the $(a, b, 0)$ class. We have

$$p_k^T = \begin{cases} 0, & k = 0 \\ \dfrac{p_k}{1 - p_0}, & k = 1, 2, \dots. \end{cases}$$

Moreover, let $P(k)$ denotes the cumulative distribution function of a member of the $(a, b, 0)$ class. Then the cdf $P^T(k)$ of the corresponding zero-truncated distribution is

$$P^T(k) = \frac{P(k) - P(0)}{1 - P(0)} = \frac{P(k) - p_0}{1 - p_0}$$

for all $k = 0, 1, 2, \dots$. Alternatively, the survival function $\bar{P}^T(k) = 1 - P^T(k)$ is

$$\bar{P}^T(k) = \frac{\bar{P}(k)}{\bar{P}(0)} = \frac{\bar{P}(k)}{1 - p_0}.$$

Package **actuar** provides `d`, `p`, `q` and `r` functions for the all the zero-truncated distributions mentioned above. Table 2 lists the root names of the functions; see Appendix C for additional details.

| Distribution | Root |
|---|---|
| Zero-truncated Poisson | `ztpois` |
| Zero-truncated binomial | `ztbinom` |
| Zero-truncated negative binomial | `ztnbinom` |
| Zero-truncated geometric | `ztgeom` |
| Logarithmic | `logarithmic` |
| Zero-modified Poisson | `zmpois` |
| Zero-modified binomial | `zmbinom` |
| Zero-modified negative binomial | `zmnbinom` |
| Zero-modified geometric | `zmgeom` |
| Zero-modified logarithmic | `zmlogarithmic` |

Table 2: Members of the $(a, b, 1)$ class of discrete distributions supported by **actuar** and root names of the R functions.

An entry of Table 2 deserves a few additional words. The logarithmic (or log-series) distribution with parameter $\theta$ has pmf

$$p_k = \frac{a\theta^x}{k}, \quad k = 1, 2, \dots,$$

with $a = -1/\log(1-\theta)$ and for $0 \leq \theta < 1$. This is the standard parametrization in the literature (Johnson et al., 2005).

The logarithmic distribution is always defined on the strictly positive integers. As such, it is not qualified as "zero-truncated", but it nevertheless belongs to the $(a, b, 1)$ class of distributions, more specifically to the subclass with $p_0 = 0$. Actually, the logarithmic distribution is the limiting case of the zero-truncated negative binomial distribution with size parameter equal to zero and $\theta = 1 - p$, where $p$ is the probability of success for the zero-truncated negative binomial. Note that this differs from the presentation in Klugman et al. (2012).

Another subclass of the $(a, b, 1)$ class of distributions is obtained by setting $p_0$ to some arbitrary number $p_0^M$ subject to $0 < p_0^M \leq 1$. The members of this subclass are called *zero-modified* distributions. Zero-modified distributions are discrete mixtures between a degenerate distribution at zero and the corresponding distribution from the $(a, b, 0)$ class.

Let $p_k^M$ and $P^M(k)$ denote the pmf and cdf of a zero-modified distribution. Written as a mixture, the pmf is

$$p_k^M = \left(1 - \frac{1 - p_0^M}{1 - p_0}\right) \mathbb{1}_{\{k=0\}} + \frac{1 - p_0^M}{1 - p_0}\, p_k. \tag{4}$$

Alternatively, we have

$$p_k^M = \begin{cases} p_0^M, & k = 0 \\ \dfrac{1 - p_0^M}{1 - p_0}\, p_k, & k = 1, 2, \dots \end{cases}$$

and

$$P^M(k) = p_0^M + (1 - p_0^M)\frac{P(k) - P(0)}{1 - P(0)} = p_0^M + \frac{1 - p_0^M}{1 - p_0}\,(P(k) - p_0)$$

for all $k = 0, 1, 2, \dots$. The survival function is

$$\bar{P}^M(k) = (1 - p_0^M)\,\frac{\bar{P}(k)}{\bar{P}(0)} = \frac{1 - p_0^M}{1 - p_0}\,\bar{P}(k).$$

The members of the subclass are the zero-modified Poisson, zero-modified binomial, zero-modified negative binomial and zero-modified geometric, together with the zero-modified logarithmic as a limiting case of the zero-modified negative binomial. Table 2 lists the root names of the support functions provided in **actuar**; see also Appendix C.

Quite obviously, zero-truncated distributions are zero-modified distributions with $p_0^M = 0$. However, using the dedicated functions in R will be more efficient.

# 5 Support for the Poisson-inverse Gaussian distribution

The Poisson-inverse Gaussian (PIG) distribution results from the continuous mixture between a Poisson distribution and an inverse Gaussian. That is, the Poisson-inverse Gaussian is the (marginal) distribution of the random variable $X$ when the conditional random variable $X|\Lambda = \lambda$ is Poisson with parameter $\lambda$ and the random variable $\Lambda$ is inverse Gaussian distribution with parameters $\mu$ and $\phi$.

The literature proposes many different expressions for the pmf of the PIG (Holla, 1966; Shaban, 1981; Johnson et al., 2005; Klugman et al., 2012). Using the parametrization for the inverse Gaussian found in Appendix A, we have:

$$\begin{aligned} p_x &= \sqrt{\frac{2}{\pi\phi}}\frac{e^{(\phi\mu)^{-1}}}{x!}\left(\sqrt{2\phi\left(1 + \frac{1}{2\phi\mu^2}\right)}\right)^{-(x-\frac{1}{2})} \\ &\quad \times K_{x-\frac{1}{2}}\left(\sqrt{\frac{2}{\phi}\left(1 + \frac{1}{2\phi\mu^2}\right)}\right), \end{aligned} \tag{5}$$

for $x = 0, 1, \dots$, $\mu > 0$, $\phi > 0$ and where

$$K_\nu(ax) = \frac{a^{-\nu}}{2}\int_0^\infty t^{\nu-1}e^{-z(t+at^{-1})/2}dt, \quad a^2z > 0 \tag{6}$$

is the modified Bessel function of the third kind (Bateman, 1953; Abramowitz and Stegun, 1972).

One may compute the probabilities $p_x$, $x = 0, 1, \ldots$ recursively using the following equations:

$$
\begin{aligned}
p_0 &= \exp\left\{\frac{1}{\phi\mu}\left(1 - \sqrt{1 + 2\phi\mu^2}\right)\right\} \\
p_1 &= \frac{\mu}{\sqrt{1 + 2\phi\mu^2}}\, p_0 \\
p_x &= \frac{2\phi\mu^2}{1 + 2\phi\mu^2}\left(1 - \frac{3}{2x}\right) p_{x-1} + \frac{\mu^2}{1 + 2\phi\mu^2}\,\frac{1}{x(x-1)}\, p_{x-2}, \quad x = 2, 3, \ldots.
\end{aligned}
\tag{7}
$$

The first moment of the distribution is $\mu$. The second and third central moment are, respectively,

$$
\begin{aligned}
\mu_2 &= \sigma^2 = \mu + \phi\mu^3 \\
\mu_3 &= \mu + 3\phi\mu^2\sigma^2.
\end{aligned}
$$

For the limiting case $\mu = \infty$, the underlying inverse Gaussian has an inverse chi-squared distribution. The latter has no finite strictly positive, integer moments and, consequently, neither does the Poisson-inverse Gaussian. See subsection C.4 for the formulas in this case.

# 6 Special integrals

Many of the cumulative distribution functions of Appendix A are expressed in terms of the incomplete gamma function or the incomplete beta function.

From a probability theory perspective, the incomplete gamma function is usually defined as

$$
\Gamma(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\, dt, \quad \alpha > 0, x > 0, \tag{8}
$$

with

$$
\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt,
$$

whereas the (regularized) incomplete beta function is defined as

$$
\beta(a, b; x) = \frac{1}{\beta(a, b)} \int_0^x t^{a-1} (1-t)^{b-1}\, dt, \quad a > 0, b > 0, 0 < x < 1, \tag{9}
$$

with

$$
\beta(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1}\, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.
$$

Now, there exist alternative definitions of the these functions that are valid for negative values of the parameters. Klugman et al. (2012) introduce them to extend the range of admissible values for limited expected value functions.

First, following Abramowitz and Stegun (1972, Section 6.5), we define the "extended" incomplete gamma function as

$$G(\alpha; x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt \tag{10}$$

for $\alpha$ real and $x > 0$. When $\alpha > 0$, we clearly have

$$G(\alpha; x) = \Gamma(a)[1 - \Gamma(\alpha; x)]. \tag{11}$$

The integral is also defined for $\alpha \leq 0$.

As outlined in Klugman et al. (2012, Appendix A), integration by parts of (10) yields the relation

$$G(\alpha; x) = -\frac{x^\alpha e^{-x}}{\alpha} + \frac{1}{\alpha} G(\alpha + 1; x).$$

This process can be repeated until $\alpha + k$ is a positive number, in which case the right hand side can be evaluated with (11). If $\alpha = 0, -1, -2, \dots$, this calculation requires the value of

$$G(0; x) = \int_x^\infty \frac{e^{-t}}{t}\, dt = E_1(x),$$

which is known in the literature as the *exponential integral* (Abramowitz and Stegun, 1972, Section 5.1).

Second, as seen in Abramowitz and Stegun (1972, Section 6.6), we have the following relation for the integral on the right hand side of (9):

$$\int\limits_0^x t^{a-1}(1-t)^{b-1}\, dt = \frac{x^a}{a} F(a, 1-b; a+1; x),$$

where

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^\infty \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}$$

is the Gauss hypergeometric series. With the above definition, the incomplete beta function also admits negative, non integer values for parameters $a$ and $b$.

Now, let

$$B(a, b; x) = \Gamma(a+b) \int_0^x t^{a-1}(1-t)^{b-1} dt \tag{12}$$

for $a > 0$, $b \neq -1, -2, \dots$ and $0 < x < 1$. Again, it is clear that when $b > 0$,

$$B(a, b; x) = \Gamma(a)\Gamma(b)\beta(a, b; x).$$

Of more interest here is the case where $b < 0$, $b \neq -1, -2, \dots$ and $a > 1 + \lfloor -b \rfloor$. Integration by parts of (12) yields

$$\begin{aligned} B(a,b;x) = -\Gamma(a+b) \Bigg[ & \frac{x^{a-1}(1-x)^b}{b} + \frac{(a-1)x^{a-2}(1-x)^{b+1}}{b(b+1)} \\ & + \cdots + \frac{(a-1)\cdots(a-r)x^{a-r-1}(1-x)^{b+r}}{b(b+1)\cdots(b+r)} \Bigg] \\ & + \frac{(a-1)\cdots(a-r-1)}{b(b+1)\cdots(b+r)} \Gamma(a-r-1) \\ & \times \Gamma(b+r+1)\beta(a-r-1, b+r+1; x), \end{aligned} \tag{13}$$

where $r = \lfloor -b \rfloor$. For the needs of **actuar**, we dubbed (12) the *beta integral*.

Package **actuar** includes a C implementation of (13) and imports functionalities of package **expint** (Goulet, 2017) to compute the incomplete gamma function (10) at the C level. The routines are used to evaluate the limited expected value for distributions of the transformed beta and transformed gamma families.

# 7 Implementation details

The core of all the functions presented in this document is written in C for speed.

The cdf of the continuous distributions of Table 1 use `pbeta` and `pgamma` to compute the incomplete beta and incomplete gamma functions, respectively. Functions `dinvgauss`, `pinvgauss` and `qinvgauss` rely on C implementations of functions of the same name from package **statmod** (Giner and Smyth, 2016).

The matrix exponential C routine needed in `dphtype` and `pphtype` is based on `expm` from package **Matrix** (Bates and Maechler, 2016).

The C code to compute the beta integral (13) was written by the second author.

For all but the trivial input values, the pmf, cdf and quantile functions for the zero-truncated and zero-modified distributions of Table 2 use the internal R functions for the corresponding standard distribution.

Generation of random variates from zero-truncated distributions uses the following simple inversion algorithm on a restricted range (Dalgaard, 2005; Thomopoulos, 2013). Let $u$ be a random number from a uniform distribution on $(p_0, 1)$. Then $x = P^{-1}(u)$ is distributed according to the zero-truncated version of the distribution with cdf $P(k)$.

For zero-modified distributions, we generate variates from the discrete mixture (4) when $p_0^M \geq p_0$. When $p_0^M < p_0$, we can use either of two methods:

i) the classical rejection method with an envelope that differs from the target distribution only at zero (meaning that only zeros are rejected);

ii) the inversion method on a restricted range explained above.

Which approach is faster depends on the relative speeds of the standard random generation function and the standard quantile function, and also on the proportion of zeros that are rejected using the rejection algorithm. Based on the difference $p_0 - p_0^M$, we determined (empirically) distribution-specific cutoff points between the two methods.

Finally, computation of the Poisson-inverse Gaussian pmf uses the direct expression (5) — and the C level function `bessel_k` part of the R API — rather than the recursive equations (7). We thereby take advantage of the various optimizations in `bessel_k`, with no negative impact on performance.

# A Continuous distributions

This appendix gives the root name and the parameters of the R support functions for the distributions of Table 1, as well as the formulas for the pdf, the cdf, the raw moment of order $k$ and the limited moment of order $k$ using the parametrization of Klugman et al. (2012) and Hogg and Klugman (1984).

In the following, $\Gamma(\alpha; x)$ is the incomplete gamma function (8), $\beta(a, b; x)$ is the incomplete beta function (9), $G(\alpha; x)$ is the "extended" incomplete gamma function (10), $B(a, b; x)$ is the beta integral (12) and $K_\nu(x)$ is the modified Bessel function of the third kind (6).

Unless otherwise stated, all parameters are finite and strictly positive, and the functions are defined for $x > 0$.

## A.1 Transformed beta family

### A.1.1 Transformed beta

Root: `trbeta`, `pearson6`
Parameters: `shape1` ($\alpha$), `shape2` ($\gamma$), `shape3` ($\tau$), `rate` ($\lambda = 1/\theta$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{\gamma u^\tau (1-u)^\alpha}{x \beta(\alpha, \tau)}, \qquad u = \frac{v}{1+v}, \qquad v = \left(\frac{x}{\theta}\right)^\gamma \\
F(x) &= \beta(\tau, \alpha; u) \\
E[X^k] &= \frac{\theta^k \Gamma(\tau + k/\gamma) \Gamma(\alpha - k/\gamma)}{\Gamma(\alpha)\Gamma(\tau)}, \qquad -\tau\gamma < k < \alpha\gamma \\
E[(X \wedge x)^k] &= \frac{\theta^k B(\tau + k/\gamma, \alpha - k/\gamma; u)}{\Gamma(\alpha)\Gamma(\tau)} \\
&\quad + x^k [1 - \beta(\tau, \alpha; u)], \qquad k > -\tau\gamma
\end{aligned}
$$

### A.1.2 Burr

Root: `burr`

Parameters: shape1 ($\alpha$), shape2 ($\gamma$), rate ($\lambda = 1/\theta$), scale ($\theta$)

$$\begin{aligned}
f(x) &= \frac{\alpha\gamma u^\alpha (1-u)}{x}, \qquad u = \frac{1}{1+v}, \qquad v = \left(\frac{x}{\theta}\right)^\gamma \\
F(x) &= 1 - u^\alpha \\
E[X^k] &= \frac{\theta^k \Gamma(1+k/\gamma)\Gamma(\alpha - k/\gamma)}{\Gamma(\alpha)}, \qquad -\gamma < k < \alpha\gamma \\
E[(X \wedge x)^k] &= \frac{\theta^k B(1+k/\gamma, \alpha - k/\gamma; 1-u)}{\Gamma(\alpha)} \\
&\quad + x^k u^\alpha, \qquad k > -\gamma
\end{aligned}$$

### A.1.3 Loglogistic

Root: `llogis`
Parameters: shape ($\gamma$), rate ($\lambda = 1/\theta$), scale ($\theta$)

$$\begin{aligned}
f(x) &= \frac{\gamma u (1-u)}{x}, \qquad u = \frac{v}{1+v}, \qquad v = \left(\frac{x}{\theta}\right)^\gamma \\
F(x) &= u \\
E[X^k] &= \theta^k \Gamma(1+k/\gamma)\Gamma(1 - k/\gamma), \qquad -\gamma < k < \gamma \\
E[(X \wedge x)^k] &= \theta^k B(1+k/\gamma, 1 - k/\gamma; u) \\
&\quad + x^k (1-u), \qquad k > -\gamma
\end{aligned}$$

### A.1.4 Paralogistic

Root: `paralogis`
Parameters: shape ($\alpha$), rate ($\lambda = 1/\theta$), scale ($\theta$)

$$\begin{aligned}
f(x) &= \frac{\alpha^2 u^\alpha (1-u)}{x}, \qquad u = \frac{1}{1+v}, \qquad v = \left(\frac{x}{\theta}\right)^\alpha \\
F(x) &= 1 - u^\alpha \\
E[X^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha - k/\alpha)}{\Gamma(\alpha)}, \qquad -\alpha < k < \alpha^2 \\
E[(X \wedge x)^k] &= \frac{\theta^k B(1+k/\alpha, \alpha - k/\alpha; 1-u)}{\Gamma(\alpha)} \\
&\quad + x^k u^\alpha, \qquad k > -\alpha
\end{aligned}$$

### A.1.5 Generalized Pareto

Root: `genpareto`
Parameters: shape1 ($\alpha$), shape2 ($\tau$), rate ($\lambda = 1/\theta$), scale ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{u^\tau (1-u)^\alpha}{x \beta(\alpha, \tau)}, \qquad u = \frac{v}{1+v}, \qquad v = \frac{x}{\theta} \\
F(x) &= \beta(\tau, \alpha; u) \\
E[X^k] &= \frac{\theta^k \Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}, \qquad -\tau < k < \alpha \\
E[(X \wedge x)^k] &= \frac{\theta^k B(\tau+k, \alpha-k; u)}{\Gamma(\alpha)\Gamma(\tau)} \\
&\quad + x^k [1 - \beta(\tau, \alpha; u)], \qquad k > -\tau
\end{aligned}
$$

### A.1.6 Pareto

Root: `pareto`, `pareto2`
Parameters: `shape` ($\alpha$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{\alpha u^\alpha (1-u)}{x}, \qquad u = \frac{1}{1+v}, \qquad v = \frac{x}{\theta} \\
F(x) &= 1 - u^\alpha \\
E[X^k] &= \frac{\theta^k \Gamma(1+k)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \qquad -1 < k < \alpha \\
E[(X \wedge x)^k] &= \frac{\theta^k B(1+k, \alpha-k; 1-u)}{\Gamma(\alpha)} \\
&\quad + x^k u^\alpha, \qquad k > -1
\end{aligned}
$$

### A.1.7 Inverse Burr

Root: `invburr`
Parameters: `shape1` ($\tau$), `shape2` ($\gamma$), `rate` ($\lambda = 1/\theta$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{\tau \gamma u^\tau (1-u)}{x}, \qquad u = \frac{v}{1+v}, \qquad v = \left(\frac{x}{\theta}\right)^\gamma \\
F(x) &= u^\tau \\
E[X^k] &= \frac{\theta^k \Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}, \qquad -\gamma < k < \alpha\gamma \\
E[(X \wedge x)^k] &= \frac{\theta^k B(\tau+k/\gamma, 1-k/\gamma; u)}{\Gamma(\tau)} \\
&\quad + x^k (1 - u^\tau), \qquad k > -\tau\gamma
\end{aligned}
$$

### A.1.8 Inverse Pareto

Root: `invpareto`
Parameters: `shape` ($\tau$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{\tau u^\tau (1-u)}{x}, \qquad u = \frac{v}{1+v}, \qquad v = \frac{x}{\theta} \\
F(x) &= u^\tau \\
E[X^k] &= \frac{\theta^k \Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \qquad -\tau < k < 1 \\
E[(X \wedge x)^k] &= \theta^k \tau \int_0^u y^{\tau+k-1}(1-y)^{-k}\,dy \\
&\quad + x^k(1-u^\tau), \qquad k > -\tau
\end{aligned}
$$

### A.1.9 Inverse paralogistic

Root: `invparalogis`
Parameters: `shape` ($\tau$), `rate` ($\lambda = 1/\theta$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{\tau^2 u^\tau (1-u)}{x}, \qquad u = \frac{v}{1+v}, \qquad v = \left(\frac{x}{\theta}\right)^\tau \\
F(x) &= u^\tau \\
E[X^k] &= \frac{\theta^k \Gamma(\tau+k/\tau)\Gamma(1-k/\tau)}{\Gamma(\tau)}, \qquad -\tau^2 < k < \tau \\
E[(X \wedge x)^k] &= \frac{\theta^k B(\tau+k/\tau, 1-k/\tau; u)}{\Gamma(\tau)} \\
&\quad + x^k(1-u^\tau), \qquad k > -\tau^2
\end{aligned}
$$

## A.2 Transformed gamma family

### A.2.1 Transformed gamma

Root: `trgamma`
Parameters: `shape1` ($\alpha$), `shape2` ($\tau$), `rate` ($\lambda = 1/\theta$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{\tau u^\alpha e^{-u}}{x\Gamma(\alpha)}, \qquad u = \left(\frac{x}{\theta}\right)^\tau \\
F(x) &= \Gamma(\alpha; u) \\
E[X^k] &= \frac{\theta^k \Gamma(\alpha+k/\tau)}{\Gamma(\alpha)} \qquad k > -\alpha\tau \\
E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha+k/\tau)}{\Gamma(\alpha)} \Gamma(\alpha+k/\tau; u) \\
&\quad + x^k[1-\Gamma(\alpha; u)], \qquad k > -\alpha\tau
\end{aligned}
$$

### A.2.2 Inverse transformed gamma

Root: `invtrgamma`
Parameters: `shape1` ($\alpha$), `shape2` ($\tau$), `rate` ($\lambda = 1/\theta$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{\tau u^\alpha e^{-u}}{x\Gamma(\alpha)}, \qquad u = \left(\frac{\theta}{x}\right)^\tau \\
F(x) &= 1 - \Gamma(\alpha; u) \\
E[X^k] &= \frac{\theta^k \Gamma(\alpha - k/\tau)}{\Gamma(\alpha)} \qquad k < \alpha\tau \\
E[(X \wedge x)^k] &= \frac{\theta^k G(\alpha - k/\tau; u)}{\Gamma(\alpha)} + x^k \Gamma(\alpha; u), \qquad \text{all } k
\end{aligned}
$$

### A.2.3 Inverse gamma

Root: `invgamma`
Parameters: `shape` ($\alpha$), `rate` ($\lambda = 1/\theta$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{u^\alpha e^{-u}}{x\Gamma(\alpha)}, \qquad u = \frac{\theta}{x} \\
F(x) &= 1 - \Gamma(\alpha; u) \\
E[X^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)} \qquad k < \alpha \\
E[(X \wedge x)^k] &= \frac{\theta^k G(\alpha - k; u)}{\Gamma(\alpha)} + x^k \Gamma(\alpha; u), \qquad \text{all } k \\
M(t) &= \frac{2}{\Gamma(\alpha)} (-\theta t)^{\alpha/2} K_\alpha(\sqrt{-4\theta t})
\end{aligned}
$$

### A.2.4 Inverse Weibull

Root: `invweibull`, `lgompertz`
Parameters: `shape` ($\tau$), `rate` ($\lambda = 1/\theta$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{\tau u e^{-u}}{x}, \qquad u = \left(\frac{\theta}{x}\right)^\tau \\
F(x) &= e^{-u} \\
E[X^k] &= \theta^k \Gamma(1 - k/\tau) \qquad k < \tau \\
E[(X \wedge x)^k] &= \theta^k G(1 - k/\tau; u) + x^k (1 - e^{-u}), \qquad \text{all } k
\end{aligned}
$$

### A.2.5 Inverse exponential

Root: `invexp`
Parameters: `rate` ($\lambda = 1/\theta$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{u e^{-u}}{x}, \qquad u = \frac{\theta}{x} \\
F(x) &= e^{-u} \\
E[X^k] &= \theta^k \Gamma(1 - k) \qquad k < 1
\end{aligned}
$$

$$E[(X \wedge x)^k] = \theta^k G(1-k; u) + x^k(1-e^{-u}), \qquad \text{all } k$$

## A.3 Other distributions

### A.3.1 Loggamma

Root: `lgamma`
Parameters: `shapelog` ($\alpha$), `ratelog` ($\lambda$)

$$\begin{aligned}
f(x) &= \frac{\lambda^\alpha (\ln x)^{\alpha-1}}{x^{\lambda+1}\Gamma(\alpha)}, \qquad x > 1 \\
F(x) &= \Gamma(\alpha; \lambda \ln x), \qquad x > 1 \\
E[X^k] &= \left(\frac{\lambda}{\lambda - x}\right)^\alpha, \qquad k < \lambda \\
E[(X \wedge x)^k] &= \left(\frac{\lambda}{\lambda - x}\right)^\alpha \Gamma(\alpha; (\lambda - k)\ln x) \\
&\quad + x^k \Gamma(\alpha; \lambda \ln x), \qquad k < \lambda
\end{aligned}$$

### A.3.2 Gumbel

Root: `gumbel`
Parameters: `alpha` ($-\infty < \alpha < \infty$), `scale` ($\theta$)

$$\begin{aligned}
f(x) &= \frac{e^{-(u+e^{-u})}}{\theta}, \qquad u = \frac{x-\alpha}{\theta}, \qquad -\infty < x < \infty \\
F(x) &= \exp[-\exp(-u)] \\
E[X] &= \alpha + \gamma\theta, \qquad \gamma \approx 0.57721566490153 \\
\mathrm{Var}[X] &= \frac{\pi^2\theta^2}{6} \\
M(t) &= e^{\alpha t}\Gamma(1-\theta t)
\end{aligned}$$

### A.3.3 Inverse Gaussian

Root: `invgauss`
Parameters: `mean` ($\mu$), `shape` ($\lambda = 1/\phi$), `dispersion` ($\phi$)

$$\begin{aligned}
f(x) &= \left(\frac{1}{2\pi\phi x^3}\right)^{1/2} \exp\left\{-\frac{(x/\mu - 1)^2}{2\phi\mu^2 x}\right\} \\
F(x) &= \Phi\left(\frac{x/\mu - 1}{\sqrt{\phi x}}\right) + e^{2/(\phi\mu)}\Phi\left(-\frac{x/\mu + 1}{\sqrt{\phi x}}\right) \\
E[X^k] &= \mu^k \sum_{i=0}^{k-1} \frac{(k+i-1)!}{i!(k-i-1)!}\left(\frac{\phi\mu}{2}\right)^i
\end{aligned}$$

$$
\begin{aligned}
E[X \wedge x] &= \mu \left[ \Phi\left( \frac{x/\mu - 1}{\sqrt{\phi x}} \right) - e^{2/(\phi\mu)} \Phi\left( -\frac{x/\mu + 1}{\sqrt{\phi x}} \right) \right] \\
&\quad + x(1 - F(x)) \\
M(t) &= \exp\left\{ \frac{1}{\phi\mu} \left( 1 - \sqrt{1 - 2\phi\mu^2 t} \right) \right\}, \qquad t \leq \frac{1}{2\phi\mu^2}
\end{aligned}
$$

The limiting case $\mu = \infty$ is an inverse gamma distribution with $\alpha = 1/2$ and $\lambda = 2\phi$ (or inverse chi-squared).

### A.3.4 Single parameter Pareto

Root: `pareto1`
Parameters: `shape` ($\alpha$), `min` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \qquad x > \theta \\
F(x) &= 1 - \left( \frac{\theta}{x} \right)^\alpha, \qquad x > \theta \\
E[X^k] &= \frac{\alpha\theta^k}{\alpha - k}, \qquad k < \alpha \\
E[(X \wedge x)^k] &= \frac{\alpha\theta^k}{\alpha - k} - \frac{k\theta^\alpha}{(\alpha - k)x^{\alpha-k}}, \qquad x \geq \theta
\end{aligned}
$$

Although there appears to be two parameters, only $\alpha$ is a true parameter. The value of $\theta$ is the minimum of the distribution and is usually set in advance.

### A.3.5 Generalized beta

Root: `genbeta`
Parameters: `shape1` ($a$), `shape2` ($b$), `shape3` ($\tau$), `rate` ($\lambda = 1/\theta$), `scale` ($\theta$)

$$
\begin{aligned}
f(x) &= \frac{\tau u^a (1 - u)^{b-1}}{x\beta(a, b)}, \qquad u = \left( \frac{x}{\theta} \right)^\tau, \qquad 0 < x < \theta \\
F(x) &= \beta(a, b; u) \\
E[X^k] &= \frac{\theta^k \beta(a + k/\tau, b)}{\beta(a, b)}, \qquad k > -a\tau \\
E[(X \wedge x)^k] &= \frac{\theta^k \beta(a + k/\tau, b)}{\beta(a, b)} \beta(a + k/\tau, b; u) \\
&\quad + x^k [1 - \beta(a, b; u)], \qquad k > -\tau\gamma
\end{aligned}
$$

# B Phase-type distributions

Consider a continuous-time Markov process with $m$ transient states and one absorbing state. Let

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{0} & 0 \end{bmatrix} \tag{14}$$

be the transition rates matrix (or intensity matrix) of such a process and let $(\boldsymbol{\pi}, \pi_{m+1})$ be the initial probability vector. Here, $\boldsymbol{T}$ is an $m \times m$ non-singular matrix with $t_{ii} < 0$ for $i = 1, \dots, m$ and $t_{ij} \geq 0$ for $i \neq j$; $\boldsymbol{\pi}$ is an $1 \times m$ vector of probabilities such that $\boldsymbol{\pi}\boldsymbol{e} + \pi_{m+1} = 1$; $\boldsymbol{t} = -\boldsymbol{T}\boldsymbol{e}$; $\boldsymbol{e} = [1]_{m \times 1}$ is a column vector of ones.

Root: `phtype`
Parameters: `prob` ($\boldsymbol{\pi}_{1 \times m}$), `rates` ($\boldsymbol{T}_{m \times m}$)

$$\begin{aligned}
f(x) &= \begin{cases} 1 - \boldsymbol{\pi}\boldsymbol{e} & x = 0, \\ \boldsymbol{\pi} e^{\boldsymbol{T}x} \boldsymbol{t}, & x > 0 \end{cases} \\
F(x) &= \begin{cases} 1 - \boldsymbol{\pi}\boldsymbol{e}, & x = 0, \\ 1 - \boldsymbol{\pi} e^{\boldsymbol{T}x} \boldsymbol{e}, & x > 0 \end{cases} \\
E[X^k] &= k! \boldsymbol{\pi} (-\boldsymbol{T})^{-k} \boldsymbol{e} \\
M(t) &= \boldsymbol{\pi} (-t\boldsymbol{I} - \boldsymbol{T})^{-1} \boldsymbol{t} + (1 - \boldsymbol{\pi}\boldsymbol{e})
\end{aligned}$$

# C Discrete distributions

This appendix gives the root name and the parameters of the R support functions for the members of the $(a, b, 0)$ and $(a, b, 1)$ discrete distributions as defined in Klugman et al. (2012); the values of $a$, $b$ and $p_0$ in the representation; the pmf; the relationship with other distributions, when there is one. The appendix also provides the main characteristics of the Poisson-inverse Gaussian distribution.

## C.1 The $(a, b, 0)$ class

The distributions in this section are all supported in base R. Their pmf can be computed recursively by fixing $p_0$ to the specified value and then using $p_k = (a + b/k) p_{k-1}$, for $k = 1, 2, \dots$.

All parameters are finite.

### C.1.1 Poisson

Root: `pois`
Parameter: `lambda` ($\lambda \geq 0$)

$$a = 0, \qquad b = \lambda, \qquad p_0 = e^{-\lambda}$$
$$p_k = \frac{e^{-\lambda}\lambda^k}{k!}$$

### C.1.2 Negative binomial

Root: `nbinom`
Parameters: `size` ($r \geq 0$), `prob` ($0 < p \leq 1$), `mu` ($r(1-p)/p$)

$$a = 1 - p, \qquad b = (r-1)(1-p), \qquad p_0 = p^r$$
$$p_k = \binom{r+k-1}{k} p^r (1-p)^k$$

Special case: Geometric$(p)$ when $r = 1$.

### C.1.3 Geometric

Root: `geom`
Parameter: `prob` ($0 < p \leq 1$)

$$a = 1 - p, \qquad b = 0, \qquad p_0 = p$$
$$p_k = p(1-p)^k$$

### C.1.4 Binomial

Root: `binom`
Parameters: `size` ($n = 0, 1, 2, \dots$), `prob` ($0 \leq p \leq 1$)

$$a = -\frac{p}{1-p}, \qquad b = \frac{(n+1)p}{1-p}, \qquad p_0 = (1-p)^n$$
$$p_k = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 1, 2, \dots, n$$

Special case: Bernoulli$(p)$ when $n = 1$.

## C.2 The zero-truncated $(a, b, 1)$ class

Package **actuar** provides support for the distributions in this section. Zero-truncated distributions have probability at zero $p_0^T = 0$. Their pmf can be computed recursively by fixing $p_1$ to the value specified below and then using $p_k = (a + b/k)p_{k-1}$, for $k = 2, 3, \dots$. The distributions are all defined on $k = 1, 2, \dots$.

The limiting case of zero-truncated distributions when $p_1$ is infinite is a point mass in $k = 1$.

### C.2.1 Zero-truncated Poisson

Root: `ztpois`
Parameter: `lambda` ($\lambda \geq 0$)

$$a = 0, \qquad b = \lambda, \qquad p_1 = \frac{\lambda}{e^{\lambda} - 1}$$
$$p_k = \frac{\lambda^k}{k!(e^{\lambda} - 1)}$$

### C.2.2 Zero-truncated negative binomial

Root: `ztnbinom`
Parameters: `size` ($r \geq 0$), `prob` ($0 < p \leq 1$)

$$a = 1 - p, \qquad b = (r-1)(1-p), \qquad p_1 = \frac{rp^r(1-p)}{1-p^r}$$
$$p_k = \binom{r+k-1}{k} \frac{p^r(1-p)^k}{1-p^r}$$

Special cases: Logarithmic$(1-p)$ when $r = 0$; Zero-truncated geometric$(p)$ when $r = 1$.

### C.2.3 Zero-truncated geometric

Root: `ztgeom`
Parameter: `prob` ($0 < p \leq 1$)

$$a = 1 - p, \qquad b = 0, \qquad p_1 = p$$
$$p_k = p(1-p)^{k-1}$$

### C.2.4 Zero-truncated binomial

Root: `ztbinom`
Parameters: `size` ($n = 0, 1, 2, \dots$), `prob` ($0 \leq p \leq 1$)

$$a = -\frac{p}{1-p}, \qquad b = \frac{(n+1)p}{1-p}, \qquad p_1 = \frac{mp(1-p)^{n-1}}{1-(1-p)^n}$$
$$p_k = \binom{n}{k} \frac{p^k(1-p)^{n-k}}{1-(1-p)^n}, \quad k = 1, 2, \dots, n$$

### C.2.5 Logarithmic

Root: `logarithmic`
Parameter: `prob` ($0 \leq p < 1$)

$$a = p, \qquad b = -p, \qquad p_1 = -\frac{p}{\log(1-p)}$$
$$p_k = -\frac{p^k}{k\log(1-p)}$$

## C.3 The zero-modified $(a, b, 1)$ class

Package **actuar** provides support for the distributions in this section. Zero-modified distributions have an arbitrary probability at zero $p_0^M \neq p_0$, where $p_0$ is the probability at zero for the corresponding member of the $(a, b, 0)$ class. Their pmf can be computed recursively by fixing $p_1$ to the value specified below and then using $p_k = (a + b/k)p_{k-1}$, for $k = 2, 3, \dots$. The distributions are all defined on $k = 0, 1, 2, \dots$.

The limiting case of zero-modified distributions when $p_1$ is infinite is a discrete mixture between a point mass in $k = 0$ (with probability $p_0^M$) and a point mass in $k = 1$ (with probability $1 - p_0^M$).

### C.3.1 Zero-modified Poisson

Root: `zmpois`
Parameters: `lambda` ($\lambda > 0$), `p0` ($0 \leq p_0^M \leq 1$)

$$a = 0, \qquad b = \lambda, \qquad p_1 = \frac{(1 - p_0^M)\lambda}{e^\lambda - 1}$$
$$p_k = \frac{(1 - p_0^M)\lambda^k}{k!(e^\lambda - 1)}$$

### C.3.2 Zero-modified negative binomial

Root: `zmnbinom`
Parameters: `size` ($r \geq 0$), `prob` ($0 < p \leq 1$), `p0` ($0 \leq p_0^M \leq 1$)

$$a = 1 - p, \qquad b = (r-1)(1-p), \qquad p_1 = \frac{(1 - p_0^M)rp^r(1-p)}{1 - p^r}$$
$$p_k = \binom{r + k - 1}{k} \frac{(1 - p_0^M)p^r(1-p)^k}{1 - p^r}$$

Special cases: Zero-modified logarithmic$(1 - p)$ when $r = 0$; Zero-modified geometric$(p)$ when $r = 1$.

### C.3.3 Zero-modified geometric

Root: `zmgeom`
Parameters: `prob` ($0 < p \leq 1$), `p0` ($0 \leq p_0^M \leq 1$)

$$a = 1 - p, \qquad b = 0, \qquad p_1 = (1 - p_0^M)p$$
$$p_k = (1 - p_0^M)p(1-p)^{k-1}$$

### C.3.4 Zero-modified binomial

Root: `zmbinom`
Parameters: `size` ($n = 0, 1, 2, \dots$), `prob` ($0 \leq p \leq 1$), `p0` ($0 \leq p_0^M \leq 1$)

$$a = -\frac{p}{1-p}, \qquad b = \frac{(n+1)p}{1-p}, \qquad p_1^M = \frac{m(1-p_0^M)p(1-p)^{n-1}}{1-(1-p)^n}$$
$$p_k = \binom{n}{k}\frac{(1-p_0^M)p^k(1-p)^{n-k}}{1-(1-p)^n}, \quad k = 1, 2, \dots, n$$

### C.3.5 Zero-modified logarithmic

Root: `logarithmic`
Parameters: `prob` ($0 \leq p < 1$), `p0` ($0 \leq p_0^M \leq 1$)

$$a = p, \qquad b = -p, \qquad p_1 = -\frac{(1-p_0^M)p}{\log(1-p)}$$
$$p_k = -\frac{(1-p_0^M)p^k}{k\log(1-p)}$$

## C.4 Other distribution

### C.4.1 Poisson-inverse Gaussian

Root: `poisinvgauss`, `pig`
Parameters: `mean` ($\mu > 0$), `shape` ($\lambda = 1/\phi$), `dispersion` ($\phi > 0$)

$$p_x = \sqrt{\frac{2}{\pi\phi}}\frac{e^{(\phi\mu)^{-1}}}{x!}\left(\sqrt{2\phi\left(1+\frac{1}{2\phi\mu^2}\right)}\right)^{-(x-\frac{1}{2})}$$
$$\times K_{x-1/2}\left(\sqrt{\frac{2}{\phi}\left(1+\frac{1}{2\phi\mu^2}\right)}\right), \quad x = 0, 1, \dots,$$

Recursively:

$$p_0 = \exp\left\{\frac{1}{\phi\mu}\left(1-\sqrt{1+2\phi\mu^2}\right)\right\}$$
$$p_1 = \frac{\mu}{\sqrt{1+2\phi\mu^2}}\,p_0$$
$$p_x = \frac{2\phi\mu^2}{1+2\phi\mu^2}\left(1-\frac{3}{2x}\right)p_{x-1} + \frac{\mu^2}{1+2\phi\mu^2}\frac{1}{x(x-1)}\,p_{x-2}, \quad x = 2, 3, \dots.$$

In the limiting case $\mu = \infty$, the pmf reduces to

$$p_x = \sqrt{\frac{2}{\pi\phi}}\frac{1}{x!}(\sqrt{2\phi})^{-(x-\frac{1}{2})}K_{x-\frac{1}{2}}(\sqrt{2/\phi}), \quad x = 0, 1, \dots$$

and the recurrence relations become

$$\begin{aligned}
p_0 &= \exp\left\{-\sqrt{2/\phi}\right\} \\
p_1 &= \frac{1}{\sqrt{2\phi}}\,p_0 \\
p_x &= \left(1 - \frac{3}{2x}\right)p_{x-1} + \frac{1}{2\phi}\frac{1}{x(x-1)}\,p_{x-2}, \quad x = 2, 3, \dots.
\end{aligned}$$

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