

Inventory of continuous and discrete distributions provided in **actuar**

Christophe Dutang
Université du Mans

Vincent Goulet
Université Laval

Mathieu Pigeon
Université du Québec à Montréal

1 Introduction

R includes functions to compute the probability density function (pdf) or the probability mass function (pmf), the cumulative distribution function (cdf) and the quantile function, as well as functions to generate variates from a fair number of continuous and discrete distributions. For some root `foo`, the support functions are named `dfoo`, `pfoo`, `qfoo` and `rfoo`, respectively.

Package **actuar** provides `d`, `p`, `q` and `r` functions for a large number of continuous distributions useful for loss severity modeling; for phase-type distributions used in computation of ruin probabilities; for zero-truncated and zero-modified extensions of the discrete distributions commonly used in loss frequency modeling; for the heavy tailed Poisson-inverse Gaussian discrete distribution. The package also introduces support functions to compute raw moments, limited moments and the moment generating function (when it exists) of continuous distributions.

2 Additional continuous distributions

The package provides support functions for all the probability distributions found in Appendix A of [Klugman et al. \(2012\)](#) and not already present in base R, excluding the $\log-t$, but including the loggamma distribution ([Hogg and Klugman, 1984](#)). These distributions mostly fall under the umbrella of extreme value or heavy tailed distributions.

[Table 1](#) lists the distributions supported by **actuar** — using the nomenclature of [Klugman et al. \(2012\)](#) — along with the root names of the R functions.

Family	Distribution	Root
Transformed beta	Transformed beta	trbeta
	Burr	burr
	Loglogistic	llogis
	Paralogistic	paralogis
	Generalized Pareto	genpareto
	Pareto	pareto
	Inverse Burr	invburr
	Inverse Pareto	invpareto
	Inverse paralogistic	invparalogis
Transformed gamma	Transformed gamma	trgamma
	Inverse transformed gamma	invtrgamma
	Inverse gamma	invgamma
	Inverse Weibull	invweibull
	Inverse exponential	invexp
Other	Loggamma	lgamma
	Gumbel	gumbel
	Inverse Gaussian	invgauss
	Single parameter Pareto	pareto1
	Generalized beta	genbeta

Table 1: Probability distributions supported by **actuar** classified by family and root names of the R functions.

[Appendix A](#) details the formulas implemented and the name of the argument corresponding to each parameter. By default, all functions (except those for the Pareto distribution) use a rate parameter equal to the inverse of the scale parameter. This differs from [Klugman et al. \(2012\)](#) but is better in line with the functions for the gamma, exponential and Weibull distributions in base R.

In addition to the `d`, `p`, `q` and `r` functions, **actuar** introduces `m`, `lev` and `mgf` functions to compute, respectively, the theoretical raw moments

$$m_k = E[X^k],$$

the theoretical limited moments

$$E[(X \wedge x)^k] = E[\min(X, x)^k]$$

and the moment generating function

$$M_X(t) = E[e^{tX}],$$

when it exists. Every distribution of [Table 1](#) is supported, along with the following distributions of base R: beta, exponential, chi-square, gamma, log-normal, normal (no `lev`), uniform and Weibull.

The `m` and `lev` functions are especially useful for estimation methods based on the matching of raw or limited moments; see the "lossdist" vignette for their empirical counterparts. The `mgf` functions come in handy to compute the adjustment coefficient in ruin theory; see the "risk" vignette.

3 Support for phase-type distributions

In addition to the 19 distributions of [Table 1](#), the package provides support for a family of distributions deserving a separate presentation. Phase-type distributions ([Neuts, 1981](#)) are defined as the distribution of the time until absorption of continuous time, finite state Markov processes with m transient states and one absorbing state. Let

$$Q = \begin{bmatrix} T & \mathbf{t} \\ \mathbf{0} & 0 \end{bmatrix} \quad (1)$$

be the transition rates matrix (or intensity matrix) of such a process and let $(\boldsymbol{\pi}, \pi_{m+1})$ be the initial probability vector. Here, T is an $m \times m$ non-singular matrix with $t_{ii} < 0$ for $i = 1, \dots, m$ and $t_{ij} \geq 0$ for $i \neq j$, $\mathbf{t} = -T\mathbf{e}$ and \mathbf{e} is a column vector with all components equal to 1. Then the cdf of the time until absorption random variable with parameters $\boldsymbol{\pi}$ and T is

$$F(x) = \begin{cases} \pi_{m+1}, & x = 0, \\ 1 - \boldsymbol{\pi} e^{T x} \mathbf{e}, & x > 0, \end{cases} \quad (2)$$

where

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} \quad (3)$$

is the matrix exponential of matrix M .

The exponential distribution, the Erlang (gamma with integer shape parameter) and discrete mixtures thereof are common special cases of phase-type distributions.

The package provides `d`, `p`, `r`, `m` and `mgf` functions for phase-type distributions. The root is `pptype` and parameters $\boldsymbol{\pi}$ and T are named `prob` and `rates`, respectively; see also [Appendix B](#).

For the package, function `ppptype` is central to the evaluation of the ruin probabilities; see `?ruin` and the "risk" vignette.

4 Extensions to standard discrete distributions

The package introduces support functions for counting distributions commonly used in loss frequency modeling. A counting distribution is a discrete distribution defined on the non-negative integers $0, 1, 2, \dots$

Let N be the counting random variable. We denote p_k the probability that the random variable N takes the value k , that is:

$$p_k = \Pr[N = k].$$

[Klugman et al. \(2012\)](#) classify counting distributions in two main classes. First, a discrete random variable is a member of the $(a, b, 0)$ class of distributions if there exists constants a and b such that

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \quad k = 1, 2, \dots$$

The probability at zero, p_0 , is set such that $\sum_{k=0}^{\infty} p_k = 1$. The members of this class are the Poisson, the binomial, the negative binomial and its special case, the geometric. These distributions are all well supported in base R with `d`, `p`, `q` and `r` functions.

The second class of distributions is the $(a, b, 1)$ class. A discrete random variable is a member of the $(a, b, 1)$ class of distributions if there exists constants a and b such that

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \quad k = 2, 3, \dots$$

One will note that recursions start at $k = 2$ for the $(a, b, 1)$ class. Therefore, the probability at zero can be any arbitrary number $0 \leq p_0 \leq 1$.

Setting $p_0 = 0$ defines a subclass of so-called *zero-truncated* distributions. The members of this subclass are the zero-truncated Poisson, the zero-truncated binomial, the zero-truncated negative binomial and the zero-truncated geometric.

Let p_k^T denote the probability mass in k for a zero-truncated distribution. As above, p_k denotes the probability mass for the corresponding member of the $(a, b, 0)$ class. We have

$$p_k^T = \begin{cases} 0, & k = 0 \\ \frac{p_k}{1 - p_0}, & k = 1, 2, \dots \end{cases}$$

Moreover, let $P(k)$ denotes the cumulative distribution function of a member of the $(a, b, 0)$ class. Then the cdf $P^T(k)$ of the corresponding zero-truncated distribution is

$$P^T(k) = \frac{P(k) - P(0)}{1 - P(0)} = \frac{P(k) - p_0}{1 - p_0}$$

for all $k = 0, 1, 2, \dots$. Alternatively, the survival function $\bar{P}^T(k) = 1 - P^T(k)$ is

$$\bar{P}^T(k) = \frac{\bar{P}(k)}{\bar{P}(0)} = \frac{\bar{P}(k)}{1 - p_0}.$$

Package **actuar** provides `d`, `p`, `q` and `r` functions for the all the zero-truncated distributions mentioned above. [Table 2](#) lists the root names of the functions; see [Appendix C](#) for additional details.

Distribution	Root
Zero-truncated Poisson	ztpois
Zero-truncated binomial	ztbinom
Zero-truncated negative binomial	ztnbinom
Zero-truncated geometric	ztgeom
Logarithmic	logarithmic
Zero-modified Poisson	zmpois
Zero-modified binomial	zmbinom
Zero-modified negative binomial	zmnbinom
Zero-modified geometric	zmgeom
Zero-modified logarithmic	zmlogarithmic

Table 2: Members of the $(a, b, 1)$ class of discrete distributions supported by **actuar** and root names of the R functions.

An entry of Table 2 deserves a few additional words. The logarithmic (or log-series) distribution with parameter θ has pmf

$$p_k = \frac{a\theta^k}{k}, \quad k = 1, 2, \dots,$$

with $a = -1/\log(1 - \theta)$ and for $0 \leq \theta < 1$. This is the standard parametrization in the literature (Johnson et al., 2005).

The logarithmic distribution is always defined on the strictly positive integers. As such, it is not qualified as “zero-truncated”, but it nevertheless belongs to the $(a, b, 1)$ class of distributions, more specifically to the subclass with $p_0 = 0$. Actually, the logarithmic distribution is the limiting case of the zero-truncated negative binomial distribution with size parameter equal to zero and $\theta = 1 - p$, where p is the probability of success for the zero-truncated negative binomial. Note that this differs from the presentation in Klugman et al. (2012).

Another subclass of the $(a, b, 1)$ class of distributions is obtained by setting p_0 to some arbitrary number p_0^M subject to $0 < p_0^M \leq 1$. The members of this subclass are called *zero-modified* distributions. Zero-modified distributions are discrete mixtures between a degenerate distribution at zero and the corresponding distribution from the $(a, b, 0)$ class.

Let p_k^M and $P^M(k)$ denote the pmf and cdf of a zero-modified distribution. Written as a mixture, the pmf is

$$p_k^M = \left(1 - \frac{1 - p_0^M}{1 - p_0}\right) \mathbb{1}_{\{k=0\}} + \frac{1 - p_0^M}{1 - p_0} p_k. \quad (4)$$

Alternatively, we have

$$p_k^M = \begin{cases} p_0^M, & k = 0 \\ \frac{1 - p_0^M}{1 - p_0} p_k, & k = 1, 2, \dots \end{cases}$$

and

$$P^M(k) = p_0^M + (1 - p_0^M) \frac{P(k) - P(0)}{1 - P(0)} = p_0^M + \frac{1 - p_0^M}{1 - p_0} (P(k) - p_0)$$

for all $k = 0, 1, 2, \dots$. The survival function is

$$\bar{P}^T(k) = (1 - p_0^M) \frac{\bar{P}(k)}{\bar{P}(0)} = \frac{1 - p_0^M}{1 - p_0} \bar{P}(k).$$

The members of the subclass are the zero-modified Poisson, zero-modified binomial, zero-modified negative binomial and zero-modified geometric, together with the zero-modified logarithmic as a limiting case of the zero-modified negative binomial. [Table 2](#) lists the root names of the support functions provided in **actuar**; see also [Appendix C](#).

Quite obviously, zero-truncated distributions are zero-modified distributions with $p_0^M = 0$. However, using the dedicated functions in R will be more efficient.

5 Support for the Poisson-inverse Gaussian distribution

The Poisson-inverse Gaussian distribution results from the continuous mixture between a Poisson distribution and an inverse Gaussian. That is, the Poisson-inverse Gaussian is the (marginal) distribution of the random variable X when the conditional random variable $X|\Lambda = \lambda$ is Poisson with parameter λ) and the random variable Λ is inverse Gaussian distribution with parameters μ and ϕ .

The literature proposes many different expressions for the pmf of the PIG ([Holla, 1966](#); [Shaban, 1981](#); [Johnson et al., 2005](#); [Klugman et al., 2012](#)). Using the parametrization for the inverse Gaussian found in [Appendix A](#), we have:

$$p_x = \sqrt{\frac{2}{\pi\phi}} \frac{e^{(\phi\mu)^{-1}}}{x!} \left(\sqrt{2\phi \left(1 + \frac{1}{2\phi\mu^2} \right)} \right)^{-(x-\frac{1}{2})} \times K_{x-\frac{1}{2}} \left(\sqrt{\frac{2}{\phi} \left(1 + \frac{1}{2\phi\mu^2} \right)} \right), \quad (5)$$

for $x = 0, 1, \dots, \mu > 0, \phi > 0$ and where

$$K_\nu(ax) = \frac{a^{-\nu}}{2} \int_0^\infty t^{\nu-1} e^{-z(t+at^{-1})/2} dt, \quad a^2 z > 0 \quad (6)$$

is the modified Bessel function of the third kind (Bateman, 1953; Abramowitz and Stegun, 1972).

One may compute the probabilities p_x , $x = 0, 1, \dots$ recursively using the following equations:

$$\begin{aligned} p_0 &= \exp \left\{ \frac{1}{\phi\mu} \left(1 - \sqrt{1 + 2\phi\mu^2} \right) \right\} \\ p_1 &= \frac{\mu}{\sqrt{1 + 2\phi\mu^2}} p_0 \\ p_x &= \frac{2\phi\mu^2}{1 + 2\phi\mu^2} \left(1 - \frac{3}{2x} \right) p_{x-1} + \frac{\mu^2}{1 + 2\phi\mu^2} \frac{1}{x(x-1)} p_{x-2}, \quad x = 2, 3, \dots \end{aligned} \quad (7)$$

The first moment of the distribution is μ . The second and third central moment are, respectively,

$$\begin{aligned} \mu_2 &= \sigma^2 = \mu + \phi\mu^3 \\ \mu_3 &= \mu + 3\phi\mu^2\sigma^2. \end{aligned}$$

For the limiting case $\mu = \infty$, the underlying inverse Gaussian has an infinite chi-squared distribution. The latter has no finite strictly positive, integer moments and, consequently, neither does the Poisson-inverse Gaussian. See subsection C.4 for the formulas in this case.

6 Special integrals

Many of the cumulative distribution functions of Appendix A are expressed in terms of the incomplete gamma function or the incomplete beta function.

The incomplete gamma function is

$$\Gamma(\alpha; x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t} dt, \quad \alpha > 0, x > 0, \quad (8)$$

with

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt,$$

whereas the (regularized) incomplete beta function is

$$\beta(a, b; x) = \frac{1}{\beta(a, b)} \int_0^x t^{a-1} (1-t)^{b-1} dt, \quad a > 0, b > 0, 0 < x < 1, \quad (9)$$

with

$$\begin{aligned} \beta(a, b) &= \int_0^1 t^{a-1} (1-t)^{b-1} dt \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \end{aligned}$$

Klugman et al. (2012) also introduce three other integrals that play a role in extending the range of admissible values for limited expected value functions.

Let

$$G(\alpha; x) = \int_x^\infty t^{\alpha-1} e^{-t} dt \quad (10)$$

for α real and $x > 0$. When $\alpha > 0$, we clearly have

$$G(\alpha; x) = \Gamma(\alpha)[1 - \Gamma(\alpha; x)]. \quad (11)$$

The integral is also defined for $\alpha \leq 0$. Integration by parts yields the relationship

$$G(\alpha; x) = -\frac{x^\alpha e^{-x}}{\alpha} + \frac{1}{\alpha} G(\alpha + 1; x).$$

This process can be repeated until $\alpha + k$ is a positive number, in which case the right hand side can be evaluated with (11). If $\alpha = 0, -1, -2, \dots$, this calculation requires the value of

$$G(0; x) = \int_x^\infty \frac{e^{-t}}{t} dt = E_1(x),$$

which is known in the literature as the *exponential integral* (Abramowitz and Stegun, 1972).

Neither Klugman et al. (2012) nor Abramowitz and Stegun (1972) provide a name for the integral (10). For the needs of the package, we dubbed it the *gamma integral*.

Let also

$$B(a, b; x) = \Gamma(a + b) \int_0^x t^{a-1} (1-t)^{b-1} dt \quad (12)$$

for $a > 0, b \neq -1, -2, \dots$ and $0 < x < 1$. Again, it is clear that when $b > 0$,

$$B(a, b; x) = \Gamma(a)\Gamma(b)\beta(a, b; x).$$

Of more interest here is the case where $b < 0, b \neq -1, -2, \dots$ and $a > 1 + \lfloor -b \rfloor$. Integration by parts of (12) yields

$$\begin{aligned} B(a, b; x) = & -\Gamma(a + b) \left[\frac{x^{a-1}(1-x)^b}{b} + \frac{(a-1)x^{a-2}(1-x)^{b+1}}{b(b+1)} \right. \\ & \left. + \dots + \frac{(a-1) \dots (a-r)x^{a-r-1}(1-x)^{b+r}}{b(b+1) \dots (b+r)} \right] \\ & + \frac{(a-1) \dots (a-r-1)}{b(b+1) \dots (b+r)} \Gamma(a-r-1) \\ & \times \Gamma(b+r+1)\beta(a-r-1, b+r+1), \end{aligned}$$

where $r = \lfloor -b \rfloor$. This expression is little found in the literature outside of Klugman et al. (2012). For the needs of **actuar**, we dubbed (12) the *beta integral*.

The package contains functions to compute the gamma, exponential and beta integrals. They are mostly used at the C level to evaluate the limited expected value for distributions of the transformed beta and transformed gamma families. The package also provides the R interfaces `gammaint`, `expint` and `betaint`, however these functions are not exported.

7 Implementation details

The core of all the functions presented in this document is written in C for speed.

The cdf of the continuous distributions of [Table 1](#) use `pbeta` and `pgamma` to compute the incomplete beta and incomplete gamma functions, respectively. Functions `dinvgauss`, `pinvgauss` and `qinvgauss` rely on C implementations of functions of the same name from package `statmod` ([Giner and Smyth, 2016](#)).

The matrix exponential C routine needed in `dphtype` and `pphtype` is based on `expm` from package `Matrix` ([Bates and Maechler, 2016](#)).

The C implementation of `expint` is based on code from the GNU Software Library ([Galassi et al.](#)). The C code for `gammaint` and `betaint` was written by the second author.

For all but the trivial input values, the `pmf`, `cdf` and `quantile` functions for the zero-truncated and zero-modified distributions of [Table 2](#) use the internal R functions for the corresponding standard distribution.

Generation of random variates from zero-truncated distributions uses the following simple inversion algorithm on a restricted range ([Dalgaard, 2005](#); [Thomopoulos, 2013](#)). Let u be a random number from a uniform distribution on $(p_0, 1)$. Then $x = P^{-1}(u)$ is distributed according to the zero-truncated version of the distribution with cdf $P(k)$.

For zero-modified distributions, we generate variates from the discrete mixture (4) when $p_0^M \geq p_0$. When $p_0^M < p_0$, we can use either of two methods:

- i) the classical rejection method with an envelope that differs from the target distribution only at zero (meaning that only zeros are rejected);
- ii) the inversion method on a restricted range explained above.

Which approach is faster depends on the relative speeds of the standard random generation function and the standard quantile function, and also on the proportion of zeros that are rejected using the rejection algorithm. Based on the difference $p_0 - p_0^M$, we determined (empirically) distribution-specific cut-off points between the two methods.

Finally, computation of the Poisson-inverse Gaussian pmf uses the direct expression (5) — and the C level function `bessel_k` part of R's API — rather than the recursive equations (7). We thereby take advantage of the various optimizations in `bessel_k`, with no negative impact on performance.

A Continuous distributions

This appendix gives the root name and the parameters of the R support functions for the distributions of [Table 1](#), as well as the formulas for the pdf, the cdf, the raw moment of order k and the limited moment of order k using the parametrization of [Klugman et al. \(2012\)](#) and [Hogg and Klugman \(1984\)](#).

In the following, $\Gamma(\alpha; x)$ is the incomplete gamma function [\(8\)](#), $\beta(a, b; x)$ is the incomplete beta function [\(9\)](#), $G(\alpha; x)$ is the gamma integral [\(10\)](#), $B(a, b; x)$ is the beta integral [\(12\)](#) and $K_\nu(x)$ is the modified Bessel function of the third kind [\(6\)](#).

Unless otherwise stated, all parameters are finite and strictly positive, and the functions are defined for $x > 0$.

A.1 Transformed beta family

A.1.1 Transformed beta

Root: trbeta, pearson6

Parameters: shape1 (α), shape2 (γ), shape3 (τ), rate ($\lambda = 1/\theta$), scale (θ)

$$f(x) = \frac{\gamma u^\tau (1-u)^\alpha}{x \beta(\alpha, \tau)}, \quad u = \frac{v}{1+v}, \quad v = \left(\frac{x}{\theta}\right)^\gamma$$

$$F(x) = \beta(\tau, \alpha; u)$$

$$E[X^k] = \frac{\theta^k \Gamma(\tau + k/\gamma) \Gamma(\alpha - k/\gamma)}{\Gamma(\alpha) \Gamma(\tau)}, \quad -\tau\gamma < k < \alpha\gamma$$

$$E[(X \wedge x)^k] = \frac{\theta^k B(\tau + k/\gamma, \alpha - k/\gamma; u)}{\Gamma(\alpha) \Gamma(\tau)} + x^k [1 - \beta(\tau, \alpha; u)], \quad k > -\tau\gamma$$

A.1.2 Burr

Root: burr

Parameters: shape1 (α), shape2 (γ), rate ($\lambda = 1/\theta$), scale (θ)

$$f(x) = \frac{\alpha \gamma u^\alpha (1-u)}{x}, \quad u = \frac{1}{1+v}, \quad v = \left(\frac{x}{\theta}\right)^\gamma$$

$$F(x) = 1 - u^\alpha$$

$$E[X^k] = \frac{\theta^k \Gamma(1 + k/\gamma) \Gamma(\alpha - k/\gamma)}{\Gamma(\alpha)}, \quad -\gamma < k < \alpha\gamma$$

$$E[(X \wedge x)^k] = \frac{\theta^k B(1 + k/\gamma, \alpha - k/\gamma; 1-u)}{\Gamma(\alpha)} + x^k u^\alpha, \quad k > -\gamma$$

A.1.3 Loglogistic

Root: llogis

Parameters: shape (γ), rate ($\lambda = 1/\theta$), scale (θ)

$$\begin{aligned}f(x) &= \frac{\gamma u(1-u)}{x}, & u &= \frac{v}{1+v}, & v &= \left(\frac{x}{\theta}\right)^\gamma \\F(x) &= u \\E[X^k] &= \theta^k \Gamma(1+k/\gamma) \Gamma(1-k/\gamma), & -\gamma &< k < \gamma \\E[(X \wedge x)^k] &= \theta^k B(1+k/\gamma, 1-k/\gamma; u) \\&+ x^k(1-u), & k &> -\gamma\end{aligned}$$

A.1.4 Paralogistic

Root: paralogis

Parameters: shape (α), rate ($\lambda = 1/\theta$), scale (θ)

$$\begin{aligned}f(x) &= \frac{\alpha^2 u^\alpha(1-u)}{x}, & u &= \frac{1}{1+v}, & v &= \left(\frac{x}{\theta}\right)^\alpha \\F(x) &= 1 - u^\alpha \\E[X^k] &= \frac{\theta^k \Gamma(1+k/\alpha) \Gamma(\alpha - k/\alpha)}{\Gamma(\alpha)}, & -\alpha &< k < \alpha^2 \\E[(X \wedge x)^k] &= \frac{\theta^k B(1+k/\alpha, \alpha - k/\alpha; 1-u)}{\Gamma(\alpha)} \\&+ x^k u^\alpha, & k &> -\alpha\end{aligned}$$

A.1.5 Generalized Pareto

Root: genpareto

Parameters: shape1 (α), shape2 (τ), rate ($\lambda = 1/\theta$), scale (θ)

$$\begin{aligned}f(x) &= \frac{u^\tau(1-u)^\alpha}{x\beta(\alpha, \tau)}, & u &= \frac{v}{1+v}, & v &= \frac{x}{\theta} \\F(x) &= \beta(\tau, \alpha; u) \\E[X^k] &= \frac{\theta^k \Gamma(\tau+k) \Gamma(\alpha-k)}{\Gamma(\alpha) \Gamma(\tau)}, & -\tau &< k < \alpha \\E[(X \wedge x)^k] &= \frac{\theta^k B(\tau+k, \alpha-k; u)}{\Gamma(\alpha) \Gamma(\tau)} \\&+ x^k [1 - \beta(\tau, \alpha; u)], & k &> -\tau\end{aligned}$$

A.1.6 Pareto

Root: pareto, pareto2

Parameters: shape (α), scale (θ)

$$f(x) = \frac{\alpha u^\alpha (1-u)}{x}, \quad u = \frac{1}{1+v}, \quad v = \frac{x}{\theta}$$

$$F(x) = 1 - u^\alpha$$

$$E[X^k] = \frac{\theta^k \Gamma(1+k) \Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha$$

$$E[(X \wedge x)^k] = \frac{\theta^k B(1+k, \alpha-k; 1-u)}{\Gamma(\alpha)} + x^k u^\alpha, \quad k > -1$$

A.1.7 Inverse Burr

Root: `invburr`

Parameters: shape1 (τ), shape2 (γ), rate ($\lambda = 1/\theta$), scale (θ)

$$f(x) = \frac{\tau \gamma u^\tau (1-u)}{x}, \quad u = \frac{v}{1+v}, \quad v = \left(\frac{x}{\theta}\right)^\gamma$$

$$F(x) = u^\tau$$

$$E[X^k] = \frac{\theta^k \Gamma(\tau + k/\gamma) \Gamma(1 - k/\gamma)}{\Gamma(\tau)}, \quad -\gamma < k < \alpha\gamma$$

$$E[(X \wedge x)^k] = \frac{\theta^k B(\tau + k/\gamma, 1 - k/\gamma; u)}{\Gamma(\tau)} + x^k (1 - u^\tau), \quad k > -\tau\gamma$$

A.1.8 Inverse Pareto

Root: `invpareto`

Parameters: shape (τ), scale (θ)

$$f(x) = \frac{\tau u^\tau (1-u)}{x}, \quad u = \frac{v}{1+v}, \quad v = \frac{x}{\theta}$$

$$F(x) = u^\tau$$

$$E[X^k] = \frac{\theta^k \Gamma(\tau + k) \Gamma(1 - k)}{\Gamma(\tau)}, \quad -\tau < k < 1$$

$$E[(X \wedge x)^k] = \theta^k \tau \int_0^u y^{\tau+k-1} (1-y)^{-k} dy + x^k (1 - u^\tau), \quad k > -\tau$$

A.1.9 Inverse paralogistic

Root: `invparalogis`

Parameters: shape (τ), rate ($\lambda = 1/\theta$), scale (θ)

$$\begin{aligned}
f(x) &= \frac{\tau^2 u^\tau (1-u)}{x}, & u &= \frac{v}{1+v}, & v &= \left(\frac{x}{\theta}\right)^\tau \\
F(x) &= u^\tau \\
E[X^k] &= \frac{\theta^k \Gamma(\tau + k/\tau) \Gamma(1 - k/\tau)}{\Gamma(\tau)}, & -\tau^2 &< k < \tau \\
E[(X \wedge x)^k] &= \frac{\theta^k B(\tau + k/\tau, 1 - k/\tau; u)}{\Gamma(\tau)} \\
&\quad + x^k (1 - u^\tau), & k &> -\tau^2
\end{aligned}$$

A.2 Transformed gamma family

A.2.1 Transformed gamma

Root: `trgamma`

Parameters: `shape1` (α), `shape2` (τ), `rate` ($\lambda = 1/\theta$), `scale` (θ)

$$\begin{aligned}
f(x) &= \frac{\tau u^\alpha e^{-u}}{x \Gamma(\alpha)}, & u &= \left(\frac{x}{\theta}\right)^\tau \\
F(x) &= \Gamma(\alpha; u) \\
E[X^k] &= \frac{\theta^k \Gamma(\alpha + k/\tau)}{\Gamma(\alpha)} & k &> -\alpha\tau \\
E[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha + k/\tau)}{\Gamma(\alpha)} \Gamma(\alpha + k/\tau; u) \\
&\quad + x^k [1 - \Gamma(\alpha; u)], & k &> -\alpha\tau
\end{aligned}$$

A.2.2 Inverse transformed gamma

Root: `invtrgamma`

Parameters: `shape1` (α), `shape2` (τ), `rate` ($\lambda = 1/\theta$), `scale` (θ)

$$\begin{aligned}
f(x) &= \frac{\tau u^\alpha e^{-u}}{x \Gamma(\alpha)}, & u &= \left(\frac{\theta}{x}\right)^\tau \\
F(x) &= 1 - \Gamma(\alpha; u) \\
E[X^k] &= \frac{\theta^k \Gamma(\alpha - k/\tau)}{\Gamma(\alpha)} & k &< \alpha\tau \\
E[(X \wedge x)^k] &= \frac{\theta^k G(\alpha - k/\tau; u)}{\Gamma(\alpha)} + x^k \Gamma(\alpha; u), & \text{all } k
\end{aligned}$$

A.2.3 Inverse gamma

Root: `invgamma`

Parameters: `shape` (α), `rate` ($\lambda = 1/\theta$), `scale` (θ)

$$\begin{aligned}
f(x) &= \frac{u^\alpha e^{-u}}{x\Gamma(\alpha)}, & u &= \frac{\theta}{x} \\
F(x) &= 1 - \Gamma(\alpha; u) \\
E[X^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)} & k < \alpha \\
E[(X \wedge x)^k] &= \frac{\theta^k G(\alpha - k; u)}{\Gamma(\alpha)} + x^k \Gamma(\alpha; u), & \text{all } k \\
M(t) &= \frac{2}{\Gamma(\alpha)} (-\theta t)^{\alpha/2} K_\alpha(\sqrt{-4\theta t})
\end{aligned}$$

A.2.4 Inverse Weibull

Root: `invweibull`, `lgompertz`

Parameters: shape (τ), rate ($\lambda = 1/\theta$), scale (θ)

$$\begin{aligned}
f(x) &= \frac{\tau u e^{-u}}{x}, & u &= \left(\frac{\theta}{x}\right)^\tau \\
F(x) &= e^{-u} \\
E[X^k] &= \theta^k \Gamma(1 - k/\tau) & k < \tau \\
E[(X \wedge x)^k] &= \theta^k G(1 - k/\tau; u) + x^k (1 - e^{-u}), & \text{all } k
\end{aligned}$$

A.2.5 Inverse exponential

Root: `invexp`

Parameters: rate ($\lambda = 1/\theta$), scale (θ)

$$\begin{aligned}
f(x) &= \frac{u e^{-u}}{x}, & u &= \frac{\theta}{x} \\
F(x) &= e^{-u} \\
E[X^k] &= \theta^k \Gamma(1 - k) & k < 1 \\
E[(X \wedge x)^k] &= \theta^k G(1 - k; u) + x^k (1 - e^{-u}), & \text{all } k
\end{aligned}$$

A.3 Other distributions

A.3.1 Loggamma

Root: `lgamma`

Parameters: `shapelog` (α), `ratelog` (λ)

$$\begin{aligned}
f(x) &= \frac{\lambda^\alpha (\ln x)^{\alpha-1}}{x^{\lambda+1} \Gamma(\alpha)}, & x > 1 \\
F(x) &= \Gamma(\alpha; \lambda \ln x), & x > 1 \\
E[X^k] &= \left(\frac{\lambda}{\lambda - x}\right)^\alpha, & k < \lambda
\end{aligned}$$

$$E[(X \wedge x)^k] = \left(\frac{\lambda}{\lambda - x}\right)^\alpha \Gamma(\alpha; (\lambda - k) \ln x) + x^k \Gamma(\alpha; \lambda \ln x), \quad k < \lambda$$

A.3.2 Gumbel

Root: gumbel

Parameters: alpha ($-\infty < \alpha < \infty$), scale (θ)

$$f(x) = \frac{e^{-(u+e^{-u})}}{\theta}, \quad u = \frac{x - \alpha}{\theta}, \quad -\infty < x < \infty$$

$$F(x) = \exp[-\exp(-u)]$$

$$E[X] = \alpha + \gamma\theta, \quad \gamma \approx 0.57721566490153$$

$$\text{Var}[X] = \frac{\pi^2\theta^2}{6}$$

$$M(t) = e^{\alpha t} \Gamma(1 - \theta t)$$

A.3.3 Inverse Gaussian

Root: invgauss

Parameters: mean (μ), shape ($\lambda = 1/\phi$), dispersion (ϕ)

$$f(x) = \left(\frac{1}{2\pi\phi x^3}\right)^{1/2} \exp\left\{-\frac{(x/\mu - 1)^2}{2\phi\mu^2 x}\right\}$$

$$F(x) = \Phi\left(\frac{x/\mu - 1}{\sqrt{\phi x}}\right) + e^{2/(\phi\mu)} \Phi\left(-\frac{x/\mu + 1}{\sqrt{\phi x}}\right)$$

$$E[X^k] = \mu^k \sum_{i=0}^{k-1} \frac{(k+i-1)!}{i!(k-i-1)!} \left(\frac{\phi\mu}{2}\right)^i$$

$$E[X \wedge x] = \mu \left[\Phi\left(\frac{x/\mu - 1}{\sqrt{\phi x}}\right) - e^{2/(\phi\mu)} \Phi\left(-\frac{x/\mu + 1}{\sqrt{\phi x}}\right) \right] + x(1 - F(x))$$

$$M(t) = \exp\left\{\frac{1}{\phi\mu} \left(1 - \sqrt{1 - 2\phi\mu^2 t}\right)\right\}, \quad t \leq \frac{1}{2\phi\mu^2}$$

The limiting case $\mu = \infty$ is an inverse gamma distribution with $\alpha = 1/2$ and $\lambda = 2\phi$ (or inverse chi-squared).

A.3.4 Single parameter Pareto

Root: pareto1

Parameters: shape (α), min (θ)

$$\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, & x > \theta \\
F(x) &= 1 - \left(\frac{\theta}{x}\right)^\alpha, & x > \theta \\
E[X^k] &= \frac{\alpha\theta^k}{\alpha - k}, & k < \alpha \\
E[(X \wedge x)^k] &= \frac{\alpha\theta^k}{\alpha - k} - \frac{k\theta^\alpha}{(\alpha - k)x^{\alpha-k}}, & x \geq \theta
\end{aligned}$$

Although there appears to be two parameters, only α is a true parameter. The value of θ is the minimum of the distribution and is usually set in advance.

A.3.5 Generalized beta

Root: genbeta

Parameters: shape1 (a), shape2 (b), shape3 (τ), rate ($\lambda = 1/\theta$), scale (θ)

$$\begin{aligned}
f(x) &= \frac{\tau u^a (1-u)^{b-1}}{x\beta(a,b)}, & u = \left(\frac{x}{\theta}\right)^\tau, & 0 < x < \theta \\
F(x) &= \beta(a,b;u) \\
E[X^k] &= \frac{\theta^k \beta(a+k/\tau, b)}{\beta(a,b)}, & k > -a\tau \\
E[(X \wedge x)^k] &= \frac{\theta^k \beta(a+k/\tau, b)}{\beta(a,b)} \beta(a+k/\tau, b;u) \\
&\quad + x^k [1 - \beta(a,b;u)], & k > -\tau\gamma
\end{aligned}$$

B Phase-type distributions

Consider a continuous-time Markov process with m transient states and one absorbing state. Let

$$\mathbf{Q} = \begin{bmatrix} \mathbf{T} & \mathbf{t} \\ \mathbf{0} & 0 \end{bmatrix}, \quad (13)$$

be the transition rates matrix (or intensity matrix) of such a process and let $(\boldsymbol{\pi}, \pi_{m+1})$ be the initial probability vector. Here, \mathbf{T} is an $m \times m$ non-singular matrix with $t_{ii} < 0$ for $i = 1, \dots, m$ and $t_{ij} \geq 0$ for $i \neq j$; $\boldsymbol{\pi}$ is an $1 \times m$ vector of probabilities such that $\boldsymbol{\pi}\mathbf{e} + \pi_{m+1} = 1$; $\mathbf{t} = -\mathbf{T}\mathbf{e}$; $\mathbf{e} = [1]_{m \times 1}$ is a column vector of ones.

Root: phtype

Parameters: prob ($\boldsymbol{\pi}_{1 \times m}$), rates ($\mathbf{T}_{m \times m}$)

$$\begin{aligned}
f(x) &= \begin{cases} 1 - \boldsymbol{\pi e} & x = 0, \\ \boldsymbol{\pi e}^{T x} \boldsymbol{t}, & x > 0 \end{cases} \\
F(x) &= \begin{cases} 1 - \boldsymbol{\pi e}, & x = 0, \\ 1 - \boldsymbol{\pi e}^{T x} \boldsymbol{e}, & x > 0 \end{cases} \\
E[X^k] &= k! \boldsymbol{\pi} (-\boldsymbol{T})^{-k} \boldsymbol{e} \\
M(\boldsymbol{t}) &= \boldsymbol{\pi} (-\boldsymbol{t I} - \boldsymbol{T})^{-1} \boldsymbol{t} + (1 - \boldsymbol{\pi e})
\end{aligned}$$

C Discrete distributions

This appendix gives the root name and the parameters of the R support functions for the members of the $(a, b, 0)$ and $(a, b, 1)$ discrete distributions as defined in [Klugman et al. \(2012\)](#); the values of a , b and p_0 in the representation; the pmf; the relationship with other distributions, when there is one. The appendix also provides the main characteristics of the Poisson-inverse Gaussian distribution.

C.1 The $(a, b, 0)$ class

The distributions in this section are all supported in base R. Their pmf can be computed recursively by fixing p_0 to the specified value and then using $p_k = (a + b/k)p_{k-1}$, for $k = 1, 2, \dots$.

All parameters are finite.

C.1.1 Poisson

Root: `pois`

Parameter: `lambda` ($\lambda \geq 0$)

$$\begin{aligned}
a &= 0, & b &= \lambda, & p_0 &= e^{-\lambda} \\
p_k &= \frac{e^{-\lambda} \lambda^k}{k!}
\end{aligned}$$

C.1.2 Negative binomial

Root: `nbinom`

Parameters: `size` ($r \geq 0$), `prob` ($0 < p \leq 1$), `mu` ($r(1-p)/p$)

$$\begin{aligned}
a &= 1 - p, & b &= (r - 1)(1 - p), & p_0 &= p^r \\
p_k &= \binom{r + k - 1}{k} p^r (1 - p)^k
\end{aligned}$$

Special case: `Geometric(p)` when $r = 1$.

C.1.3 Geometric

Root: geom

Parameter: prob ($0 < p \leq 1$)

$$a = 1 - p, \quad b = 0, \quad p_0 = p$$
$$p_k = p(1 - p)^k$$

C.1.4 Binomial

Root: binom

Parameters: size ($n = 0, 1, 2, \dots$), prob ($0 \leq p \leq 1$)

$$a = -\frac{p}{1 - p}, \quad b = \frac{(n + 1)p}{1 - p}, \quad p_0 = (1 - p)^n$$
$$p_k = \binom{n}{k} p^k (1 - p)^{n - k}, \quad k = 1, 2, \dots, n$$

Special case: Bernoulli(p) when $n = 1$.

C.2 The zero-truncated $(a, b, 1)$ class

Package **actuar** provides support for the distributions in this section. Zero-truncated distributions have probability at zero $p_0^T = 0$. Their pmf can be computed recursively by fixing p_1 to the value specified below and then using $p_k = (a + b/k)p_{k-1}$, for $k = 2, 3, \dots$. The distributions are all defined on $k = 1, 2, \dots$.

The limiting case of zero-truncated distributions when p_1 is infinite is a point mass in $k = 1$.

C.2.1 Zero-truncated Poisson

Root: ztpois

Parameter: lambda ($\lambda \geq 0$)

$$a = 0, \quad b = \lambda, \quad p_1 = \frac{\lambda}{e^\lambda - 1}$$
$$p_k = \frac{\lambda^k}{k!(e^\lambda - 1)}$$

C.2.2 Zero-truncated negative binomial

Root: ztnbinom

Parameters: size ($r \geq 0$), prob ($0 < p \leq 1$)

$$a = 1 - p, \quad b = (r - 1)(1 - p), \quad p_1 = \frac{rp^r(1 - p)}{1 - p^r}$$

$$p_k = \binom{r + k - 1}{k} \frac{p^r(1 - p)^k}{1 - p^r}$$

Special cases: Logarithmic($1 - p$) when $r = 0$; Zero-truncated geometric(p) when $r = 1$.

C.2.3 Zero-truncated geometric

Root: `ztgeom`

Parameter: `prob` ($0 < p \leq 1$)

$$a = 1 - p, \quad b = 0, \quad p_1 = p$$

$$p_k = p(1 - p)^{k-1}$$

C.2.4 Zero-truncated binomial

Root: `ztbinom`

Parameters: `size` ($n = 0, 1, 2, \dots$), `prob` ($0 \leq p \leq 1$)

$$a = -\frac{p}{1 - p}, \quad b = \frac{(n + 1)p}{1 - p}, \quad p_1 = \frac{mp(1 - p)^{n-1}}{1 - (1 - p)^n}$$

$$p_k = \binom{n}{k} \frac{p^k(1 - p)^{n-k}}{1 - (1 - p)^n}, \quad k = 1, 2, \dots, n$$

C.2.5 Logarithmic

Root: `logarithmic`

Parameter: `prob` ($0 \leq p < 1$)

$$a = p, \quad b = -p, \quad p_1 = -\frac{p}{\log(1 - p)}$$

$$p_k = -\frac{p^k}{k \log(1 - p)}$$

C.3 The zero-modified ($a, b, 1$) class

Package `actuar` provides support for the distributions in this section. Zero-modified distributions have an arbitrary probability at zero $p_0^M \neq p_0$, where p_0 is the probability at zero for the corresponding member of the $(a, b, 0)$ class. Their pmf can be computed recursively by fixing p_1 to the value specified below and then using $p_k = (a + b/k)p_{k-1}$, for $k = 2, 3, \dots$. The distributions are all defined on $k = 0, 1, 2, \dots$.

The limiting case of zero-modified distributions when p_1 is infinite is a discrete mixture between a point mass in $k = 0$ (with probability p_0^M) and a point mass in $k = 1$ (with probability $1 - p_0^M$).

C.3.1 Zero-modified Poisson

Root: `zmpois`

Parameters: `lambda` ($\lambda > 0$), `p0` ($0 \leq p_0^M \leq 1$)

$$a = 0, \quad b = \lambda, \quad p_1 = \frac{(1 - p_0^M)\lambda}{e^\lambda - 1}$$
$$p_k = \frac{(1 - p_0^M)\lambda^k}{k!(e^\lambda - 1)}$$

C.3.2 Zero-modified negative binomial

Root: `zmnbinom`

Parameters: `size` ($r \geq 0$), `prob` ($0 < p \leq 1$), `p0` ($0 \leq p_0^M \leq 1$)

$$a = 1 - p, \quad b = (r - 1)(1 - p), \quad p_1 = \frac{(1 - p_0^M)r p^r (1 - p)}{1 - p^r}$$
$$p_k = \binom{r + k - 1}{k} \frac{(1 - p_0^M)p^r (1 - p)^k}{1 - p^r}$$

Special cases: Zero-modified logarithmic($1 - p$) when $r = 0$; Zero-modified geometric(p) when $r = 1$.

C.3.3 Zero-modified geometric

Root: `zmgeom`

Parameters: `prob` ($0 < p \leq 1$), `p0` ($0 \leq p_0^M \leq 1$)

$$a = 1 - p, \quad b = 0, \quad p_1 = (1 - p_0^M)p$$
$$p_k = (1 - p_0^M)p(1 - p)^{k-1}$$

C.3.4 Zero-modified binomial

Root: `zmbinom`

Parameters: `size` ($n = 0, 1, 2, \dots$), `prob` ($0 \leq p \leq 1$), `p0` ($0 \leq p_0^M \leq 1$)

$$a = -\frac{p}{1 - p}, \quad b = \frac{(n + 1)p}{1 - p}, \quad p_1^M = \frac{m(1 - p_0^M)p(1 - p)^{n-1}}{1 - (1 - p)^n}$$
$$p_k = \binom{n}{k} \frac{(1 - p_0^M)p^k (1 - p)^{n-k}}{1 - (1 - p)^n}, \quad k = 1, 2, \dots, n$$

C.3.5 Zero-modified logarithmic

Root: `logarithmic`

Parameters: `prob` ($0 \leq p < 1$), `p0` ($0 \leq p_0^M \leq 1$)

$$a = p, \quad b = -p, \quad p_1 = -\frac{(1 - p_0^M)p}{\log(1 - p)}$$

$$p_k = -\frac{(1 - p_0^M)p^k}{k \log(1 - p)}$$

C.4 Other distribution

C.4.1 Poisson-inverse Gaussian

Root: `poisinvgauss`, `pig`

Parameters: mean ($\mu > 0$), shape ($\lambda = 1/\phi$), dispersion ($\phi > 0$)

$$p_x = \sqrt{\frac{2}{\pi\phi}} \frac{e^{(\phi\mu)^{-1}}}{x!} \left(\sqrt{2\phi \left(1 + \frac{1}{2\phi\mu^2} \right)} \right)^{-(x-\frac{1}{2})}$$

$$\times K_{x-1/2} \left(\sqrt{\frac{2}{\phi} \left(1 + \frac{1}{2\phi\mu^2} \right)} \right), \quad x = 0, 1, \dots,$$

Recursively:

$$p_0 = \exp \left\{ \frac{1}{\phi\mu} \left(1 - \sqrt{1 + 2\phi\mu^2} \right) \right\}$$

$$p_1 = \frac{\mu}{\sqrt{1 + 2\phi\mu^2}} p_0$$

$$p_x = \frac{2\phi\mu^2}{1 + 2\phi\mu^2} \left(1 - \frac{3}{2x} \right) p_{x-1} + \frac{\mu^2}{1 + 2\phi\mu^2} \frac{1}{x(x-1)} p_{x-2}, \quad x = 2, 3, \dots$$

In the limiting case $\mu = \infty$, the pmf reduces to

$$p_x = \sqrt{\frac{2}{\pi\phi}} \frac{1}{x!} (\sqrt{2\phi})^{-(x-\frac{1}{2})} K_{x-\frac{1}{2}}(\sqrt{2/\phi}), \quad x = 0, 1, \dots$$

and the recurrence relations become

$$p_0 = \exp \left\{ -\sqrt{2/\phi} \right\}$$

$$p_1 = \frac{1}{\sqrt{2\phi}} p_0$$

$$p_x = \left(1 - \frac{3}{2x} \right) p_{x-1} + \frac{1}{2\phi} \frac{1}{x(x-1)} p_{x-2}, \quad x = 2, 3, \dots$$

References

M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover, 1972. URL <http://people.math.sfu.ca/~cbm/aands/>.

- Harry Bateman. *Higher transcendental functions*, volume 2. McGraw-Hill, 1953.
- D. Bates and M. Maechler. *Matrix: A Matrix package for R*, 2016. URL <http://cran.r-project.org/package=Matrix>.
- P. Dalgaard. simulate zero-truncated poisson distribution. r-help mailing list, May 1 2005. URL <https://stat.ethz.ch/pipermail/r-help/2005-May/070680.html>.
- M. Galassi, J. Davies, J. Theiler, B. Gough, G. Jungman, Alken P., M. Booth, F. Rossi, and R. Ulerich. *GNU Scientific Library Reference Manual*, third edition. URL <https://www.gnu.org/software/gsl/>.
- G. Giner and G. K. Smyth. **statmod**: Probability calculations for the inverse gaussian distribution. *R Journal*, 8(1):339–351, 2016. URL <https://journal.r-project.org/archive/2016-1/giner-smyth.pdf>.
- R. V. Hogg and S. A. Klugman. *Loss Distributions*. Wiley, New York, 1984. ISBN 0-4718792-9-0.
- M. S. Holla. On a poisson-inverse gaussian distribution. *Metrika*, 15:377–384, 1966.
- N. L. Johnson, A. W. Kemp, and S. Kotz. *Univariate Discrete Distributions*. Wiley, 3 edition, 2005. ISBN 978-047127246-5.
- S. A. Klugman, H. H. Panjer, and G. Willmot. *Loss Models: From Data to Decisions*. Wiley, New York, 4 edition, 2012. ISBN 978-1-118-31532-3.
- M. F. Neuts. *Matrix-geometric solutions in stochastic models: an algorithmic approach*. Dover Publications, 1981. ISBN 978-0-4866834-2-3.
- S. A. Shaban. Computation of the poisson-inverse gaussian distribution. *Communications in Statistics – Theory and Methods*, 10(14):1389–1399, 1981.
- N. T. Thomopoulos. *Essentials of Monte Carlo simulation: Statistical methods for building simulation models*. Springer, 2013. ISBN 978-146146022-0.